

POINT CONTACTS AND BOUNDARY TRIPLES

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ABSTRACT. We suggest an abstract approach for point contact problems in the framework of boundary triples. Using this approach we obtain the perturbation series for a simple eigenvalue in the discrete spectrum of the model self-adjoint extension with weak point coupling.

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1. INTRODUCTION

Let H_0 be a self-adjoint operator with an isolated simple eigenvalue λ_0 . Further let V be a bounded or unbounded self-adjoint operator such that the family of operators $H(\varkappa) := H_0 + \varkappa V$ is well-defined and self-adjoint for sufficiently small coupling constants $\varkappa \in \mathbb{R}$. If V is relatively compact with respect to H_0 , then there is a smooth function $\lambda(\varkappa)$ such that $\lambda(\varkappa)$ is a simple eigenvalue of $H(\varkappa)$ for each \varkappa and $\lim_{\varkappa \rightarrow 0} \lambda(\varkappa) = \lambda_0$ holds, cf. [K]. Since the function $\lambda(\varkappa)$ is smooth it admits a Taylor-type expansion of the form

$$(1.1) \quad \lambda(\varkappa) = \lambda_0 + a\varkappa + b\varkappa^2 + O(\varkappa^3).$$

The problem is to compute the coefficients a and b of this perturbation series in terms of the operators H_0 and V .

In a slightly modified form, similar problem appears for point contacts in quantum mechanics. Typically one considers two quantum systems which do not interact, where one of them has a simple isolated eigenvalue λ_0 . If both systems are coupled by a point contact, then the eigenvalue λ_0 can move either along the real axis or become a pole of the analytic continuation of the resolvent of the coupled system in the lower complex half-plane. In the last case one speaks of resonances. The eigenvalue case realizes if the second system has no spectrum around λ_0 while the resonance case appears if the second system has continuous spectrum around λ_0 , that is, if λ_0 is an embedded eigenvalue for the decoupled systems. If the point interaction depends on a parameter \varkappa such that for $\varkappa \rightarrow 0$ the coupled system converges to the decoupled one, then again a perturbation series is expected either for the eigenvalues or for the resonances. In the following we focus on the eigenvalue case.

Perturbation series for point interactions were perhaps first studied by B. S. Pavlov in [P84, P87] for a model of point interactions with an inner structure, where the first order coefficient a was computed. A direct sum of two three-dimensional Schrödinger operators coupled by a point contact was considered by P. Exner

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in [E91]. In this paper he was able to compute the first and second order coefficients a and b . See also related work [CCF09] on spin-dependent point interactions and [CCF10] for perturbation of eigenvalues at threshold in point contact models. A survey on the resonance case can be found in [E13], see also references therein. Point contact models are often used in other areas of mathematical physics. In [P92] a model of a small window in the screen is studied. In [P12] Maxwell and Schrödinger operators are coupled via a point contact and in [P13] a model of a three-dimensional Helmholtz resonator is constructed via point coupling.

In the following we consider an abstract point contact model and are interested in the perturbation series for its eigenvalues. In particular, let \tilde{A} and \hat{A} be two densely defined closed symmetric operators in the Hilbert spaces $\tilde{\mathcal{H}}$ and $\hat{\mathcal{H}}$, respectively, both having equal finite deficiency indices (d, d) . Let us consider the direct sum $A := \tilde{A} \oplus \hat{A}$ which is also a densely defined closed symmetric operator in $\tilde{\mathcal{H}} \oplus \hat{\mathcal{H}}$ with deficiency indices $(2d, 2d)$. Further, let $A_{[\alpha]}$ and $A_{[\beta]}$ be self-adjoint extensions of \tilde{A} and \hat{A} , respectively. The Hamiltonian of the decoupled system is given by $H_0 := A_{[\alpha]} \oplus A_{[\beta]}$, which is a self-adjoint extension of A . As usually the Hamiltonian of the point coupled system is given by another self-adjoint extension H of A , which can not be decomposed into the orthogonal sum with respect to the decomposition $\tilde{\mathcal{H}} \oplus \hat{\mathcal{H}}$. The family $H(\varkappa)$ from above is now replaced by a one-parametric family of point contacts, that means, by a family of self-adjoint extensions $H(\varkappa)$ of A .

To make the problem precise we use the framework of boundary triples. In this framework a subfamily A_Λ of self-adjoint extensions of A are labeled by Hermitian matrices Λ in \mathbb{C}^{2d} via an abstract boundary condition involving Λ . In particular, there is a Hermitian matrix Λ_0 such that $H_0 = A_{\Lambda_0}$. Let us assume that λ_0 is an isolated eigenvalue of $A_{[\beta]}$, but a resolvent point of $A_{[\alpha]}$. Moreover, let $\Lambda(\varkappa)$ be a one-parametric sufficiently regular family of Hermitian matrices in \mathbb{C}^{2d} , which converges to Λ_0 as $\varkappa \rightarrow 0$. Setting $H(\varkappa) := A_{\Lambda(\varkappa)}$ one gets a family of self-adjoint extensions $H(\varkappa)$ of A which converges in an appropriate sense to H_0 as $\varkappa \rightarrow 0$ and in the discrete spectra of the operators $H(\varkappa)$ exists a branch of the type (1.1). The goal is to compute the coefficients a and b for this branch in terms of the Taylor coefficients of $\Lambda(\varkappa)$ and the abstract Weyl function $M(\cdot)$ which is an important ingredient of the boundary triple approach. In general this problem can not be reduced to the investigation of holomorphic operator families of the types (A) or (B) in the sense of Kato which are thoroughly discussed in [K]. Only in some special cases such a reduction can be done.

We solve this problem for arbitrary $d \in \mathbb{N}$ for the first order coefficient a . In the special case of $d = 1$ we obtain first and second order coefficients a and b which is beyond Pavlov [P84, P87] and covers [E91]. In general, it would be also possible to compute b for arbitrary finite deficiency indices, however, we have not included that for the purpose to avoid tedious computations.

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2. BOUNDARY TRIPLES AND WEYL FUNCTIONS

The reader may consult with [BMN02, BGP08, DM91, DM95, MN14, S12] for the theory of boundary triples and its applications. In this note we use this concept only in the case of finite deficiency indices. Throughout this section the following hypothesis is employed.

Hypothesis I. *Let A be a closed, symmetric, densely defined operator in a Hilbert space \mathcal{H} with equal finite deficiency indices (d, d) .*

Definition 2.1. *Assume that Hypothesis I holds. The triple $\{\mathbb{C}^d, \Gamma_0, \Gamma_1\}$ with $\Gamma_0, \Gamma_1: \text{dom } A^* \rightarrow \mathbb{C}^d$ is a boundary triple for A^* if the following conditions hold: the mapping $\Gamma := (\Gamma_0, \Gamma_1)^\top$ is surjective onto \mathbb{C}^{2d} and the abstract Green's identity $(A^*f, g)_\mathcal{H} - (f, A^*g)_\mathcal{H} = (\Gamma_1 f, \Gamma_0 g)_{\mathbb{C}^d} - (\Gamma_0 f, \Gamma_1 g)_{\mathbb{C}^d}$ holds for all $f, g \in \text{dom } A^*$.*

Boundary triples is an efficient tool to parametrize self-adjoint extensions of a symmetric operator.

Proposition 2.2 [DM95, Proposition 1.4], [S12, Proposition 14.7]). *Assume that Hypothesis I holds. Let $\{\mathbb{C}^d, \Gamma_0, \Gamma_1\}$ be a boundary triple for A^* . Then for each self-adjoint extension \tilde{A} of A there is a unique self-adjoint relation Θ in \mathbb{C}^d such that $\tilde{A} = A_\Theta := A^* \upharpoonright \{f \in \text{dom } A^*: \Gamma f \in \Theta\}$.*

Remark 2.3. The self-adjoint extension $A_0 := A^* \upharpoonright \ker \Gamma_0$ is distinguished. It corresponds to the self-adjoint relation $\Theta_\infty := \left\{ \begin{pmatrix} 0 \\ h \end{pmatrix} : h \in \mathbb{C}^d \right\}$. If Θ is the graph of a Hermitian matrix Λ in \mathbb{C}^d , i.e $\Theta = \text{graph}(\Lambda)$, then one easily checks that

$$(2.1) \quad A_{\text{graph}(\Lambda)} = A_\Lambda := A^* \upharpoonright \{f \in \text{dom } A^*: \Gamma_1 f = \Lambda \Gamma_0 f\}.$$

The operator Λ is called the boundary operator with respect to the boundary triple $\{\mathbb{C}^d, \Gamma_0, \Gamma_1\}$.

One can associate γ -fields and Weyl functions with boundary triples.

Definition 2.4. *Assume that Hypothesis I holds. Let $\{\mathbb{C}^d, \Gamma_0, \Gamma_1\}$ be a boundary triple for A^* . The function $\gamma: \rho(A_0) \rightarrow \mathcal{B}(\mathbb{C}^d, \mathcal{H})$ defined as*

$$\gamma(\lambda) := (\Gamma_0 \upharpoonright \ker(A^* - \lambda))^{-1}, \quad \lambda \in \rho(A_0),$$

is called the γ -field. The function $M: \rho(A_0) \rightarrow \mathbb{C}^{d \times d}$ defined as $M(\lambda) := \Gamma_1 \gamma(\lambda)$ is called the Weyl function.

Proposition 2.5. *Assume that Hypothesis I holds. Let M be the Weyl function associated with a boundary triple $\{\mathbb{C}^d, \Gamma_0, \Gamma_1\}$ for A^* . Let the self-adjoint operator A_Λ in \mathcal{H} be as in (2.1). Then the following statements hold.*

- (i) *The function $M(\cdot)$ is holomorphic on $\rho(A_0)$.*
- (ii) *For $\lambda \in \rho(A_0)$ the relation $\dim \ker(A_\Lambda - \lambda) = \dim \ker(\Lambda - M(\lambda))$ holds.*
- (iii) *If $\lambda_0 \in \rho(A_0)$ is a simple eigenvalue of A_Λ , then the function*

$$D_\Lambda(\lambda) := \det(\Lambda - M(\lambda)), \quad \lambda \in \rho(A_0),$$

has a simple zero at $\lambda = \lambda_0$. In particular, $D'_\Lambda(\lambda_0) \neq 0$ holds.

Proof. All the statements of this proposition are known. Item (i) can be found in [BGP08, Proposition 1.21], see also [S12, Proposition 14.15 (iv)] and item (ii) is given in [BGP08, Theorem 1.36 (1)], see also [S12, Proposition 14.17 (ii)]. For item (iii) see [MN14, Corollary 4.4, Proposition 5.1 (iii)]. \square

3. ABSTRACT POINT CONTACT AND ITS WEAK COUPLING REGIME

In this section we present an abstract treatment of point contacts in the framework of boundary triples and obtain the perturbation series of the simple eigenvalue in the weak coupling regime. We make use of the following hypothesis.

Hypothesis II. *Let $\tilde{\mathcal{H}}$ and $\hat{\mathcal{H}}$ be separable Hilbert spaces. Let \tilde{A} and \hat{A} be closed, densely defined, symmetric operators in $\tilde{\mathcal{H}}$ and $\hat{\mathcal{H}}$, respectively, both with deficiency indices (d, d) . Let $\{\mathbb{C}^d, \tilde{\Gamma}_0, \tilde{\Gamma}_1\}$ and $\{\mathbb{C}^d, \hat{\Gamma}_0, \hat{\Gamma}_1\}$ be boundary triples for \tilde{A}^* and \hat{A}^* , respectively.*

The next lemma appears to be useful in what follows.

Lemma 3.1 [BGP08, Section 1.4.4]. *Assume that Hypothesis II holds. Then the operator $\tilde{A} \oplus \hat{A}$ is closed, densely defined and symmetric in the Hilbert space $\tilde{\mathcal{H}} \oplus \hat{\mathcal{H}}$ with deficiency indices $(2d, 2d)$ and $\{\mathbb{C}^{2d}, \tilde{\Gamma}_0 \oplus \hat{\Gamma}_0, \tilde{\Gamma}_1 \oplus \hat{\Gamma}_1\}$ is a boundary triple for $(\tilde{A} \oplus \hat{A})^*$.*

Our model operator A_Λ in the Hilbert space $\tilde{\mathcal{H}} \oplus \hat{\mathcal{H}}$ is defined as

$$A_\Lambda(\tilde{f} \oplus \hat{f}) := \tilde{A}^* \tilde{f} \oplus \hat{A}^* \hat{f},$$

$$\text{dom } A_\Lambda := \left\{ \tilde{f} \oplus \hat{f} \in \text{dom } \tilde{A}^* \oplus \text{dom } \hat{A}^* : \Lambda \begin{pmatrix} \tilde{\Gamma}_0 \tilde{f} \\ \hat{\Gamma}_0 \hat{f} \end{pmatrix} = \begin{pmatrix} \tilde{\Gamma}_1 \tilde{f} \\ \hat{\Gamma}_1 \hat{f} \end{pmatrix} \right\},$$

with a Hermitian $2d \times 2d$ matrix of the form

$$\Lambda := \begin{pmatrix} \alpha I_d & \omega I_d \\ \bar{\omega} I_d & \beta I_d \end{pmatrix}, \quad \alpha, \beta \in \mathbb{R}, \quad \omega \in \mathbb{C}.$$

Proposition 3.2. *The operator A_Λ , defined as above, is self-adjoint in the Hilbert space $\tilde{\mathcal{H}} \oplus \hat{\mathcal{H}}$.*

Proof. The statement of this proposition is a straightforward consequence of the structure of the matrix Λ , Proposition 2.2, Remark 2.3 and Lemma 3.1. \square

The next theorem contains the main results of this note: the two terms expansion of a bound state of A_Λ for small coupling parameter $|\omega|$ in the case of arbitrary $d \in \mathbb{N}$ and the three terms analogous expansion in the special case $d = 1$. In its formulation we use self-adjoint operators

$$(3.1) \quad \begin{aligned} \tilde{A}_{[\alpha]} &:= \tilde{A}^* \upharpoonright \ker(\tilde{\Gamma}_1 - \alpha \tilde{\Gamma}_0), & \tilde{A}_0 &:= \tilde{A}^* \upharpoonright \ker \tilde{\Gamma}_0, \\ \hat{A}_{[\beta]} &:= \hat{A}^* \upharpoonright \ker(\hat{\Gamma}_1 - \beta \hat{\Gamma}_0), & \hat{A}_0 &:= \hat{A}^* \upharpoonright \ker \hat{\Gamma}_0. \end{aligned}$$

Let L be a $d \times d$ -matrix. In the following we use the notion of the adjugate matrix $\text{adj}(L)$, cf. [B08, MN99]. Notice that the adjugate of a matrix is quite different from the adjoint one L^* .

Theorem 3.3. *Assume that Hypothesis II holds with some $d \in \mathbb{N}$. Let \widetilde{M} and \widehat{M} be the Weyl functions associated with boundary triples from that hypothesis. Let the self-adjoint operators $\widetilde{A}_{[\alpha]}$ and $\widehat{A}_{[\beta]}$ be as above. Assume that the real value λ_0 satisfies $\lambda_0 \in \rho(\widetilde{A}_0) \cap \rho(\widehat{A}_0) \cap \rho(\widetilde{A}_{[\alpha]})$ and λ_0 is a simple isolated eigenvalue of $\widehat{A}_{[\beta]}$.*

(i) *Then for sufficiently small $|\omega|$ in the discrete spectrum of A_Λ there is a branch*

$$(3.2) \quad \lambda(|\omega|^2) = \lambda_0 + a|\omega|^2 + O(|\omega|^4), \quad |\omega| \rightarrow 0+,$$

with

$$(3.3) \quad a := \frac{\operatorname{tr} \left(\operatorname{adj} (\beta I_d - \widehat{M}(\lambda_0)) (\widetilde{M}(\lambda_0) - \alpha I_d)^{-1} \right)}{\operatorname{tr} \left(\operatorname{adj} (\beta I_d - \widehat{M}(\lambda_0)) \widehat{M}'(\lambda_0) \right)},$$

where $\operatorname{adj} (\beta I_d - \widehat{M}(\lambda_0))$ is the adjugate matrix.

(ii) *Suppose that $d = 1$. Then the expansion (3.2) can be extended as*

$$(3.4) \quad \lambda(|\omega|^2) = \lambda_0 + a|\omega|^2 + b|\omega|^4 + O(|\omega|^6), \quad |\omega| \rightarrow 0+,$$

with

$$(3.5) \quad a := \frac{1}{(\widetilde{M}(\lambda_0) - \alpha) \widehat{M}'(\lambda_0)}$$

and

$$(3.6) \quad b := \frac{1}{((\widetilde{M}(\lambda_0) - \alpha) \widehat{M}'(\lambda_0))^2} \left(\frac{\widetilde{M}'(\lambda_0)}{\alpha - \widetilde{M}(\lambda_0)} - \frac{1}{2} \frac{\widehat{M}''(\lambda_0)}{\widehat{M}'(\lambda_0)} \right).$$

Proof. (i) The proof of this item is carried out in three steps.

Step I. For sufficiently small $\varepsilon > 0$ the interval $I := (\lambda_0 - \varepsilon, \lambda_0 + \varepsilon)$ is contained in the set $\rho(\widetilde{A}_0) \cap \rho(\widehat{A}_0)$. By Proposition 2.5 (i) the following matrix-valued function

$$T(\lambda) := (\alpha I_d - \widetilde{M}(\lambda))(\beta I_d - \widehat{M}(\lambda))$$

is well-defined and C^∞ -smooth on I . Next we introduce the scalar-valued function

$$(3.7) \quad F: I \times \mathbb{R} \rightarrow \mathbb{R}, \quad F(\lambda, x) := \det (T(\lambda) - x I_d),$$

which is C^∞ -smooth on $I \times \mathbb{R}$.

Step II. The following two functions

$$\widetilde{D}_\alpha(\lambda) := \det (\alpha I_d - \widetilde{M}(\lambda)) \quad \text{and} \quad \widehat{D}_\beta(\lambda) := \det (\beta I_d - \widehat{M}(\lambda))$$

are well-defined and C^∞ -smooth on I . Jacobi's formula [G72, MN99] and the identity $\operatorname{adj} (L_1 L_2) = \operatorname{adj} (L_2) \operatorname{adj} (L_1)$ imply

$$F_x(\lambda_0, 0) = -\operatorname{tr} \left(\operatorname{adj} (\beta I_d - \widehat{M}(\lambda_0)) \operatorname{adj} (\alpha I_d - \widetilde{M}(\lambda_0)) \right).$$

In view of $\lambda_0 \in \rho(\widetilde{A}_{[\alpha]})$ and of Proposition 2.5 (ii) the matrix $\alpha I_d - \widetilde{M}(\lambda_0)$ is invertible. For any invertible matrix L the identity $\operatorname{adj} (L) = \det (L) L^{-1}$ holds. Hence, we arrive at

$$(3.8) \quad F_x(\lambda_0, 0) = \widetilde{D}_\alpha(\lambda_0) \operatorname{tr} \left(\operatorname{adj} (\beta I_d - \widehat{M}(\lambda_0)) (\widetilde{M}(\lambda_0) - \alpha I_d)^{-1} \right).$$

Note that $F(\lambda, 0) = \tilde{D}_\alpha(\lambda)\widehat{D}_\beta(\lambda)$, where the identity $\det(L_1 L_2) = \det(L_1)\det(L_2)$ is used. In view of $\lambda_0 \in \sigma_d(\widehat{A}_{[\beta]})$ and of Proposition 2.5(ii) we get $\widehat{D}_\beta(\lambda_0) = 0$, which implies $F(\lambda_0, 0) = 0$. Next we compute F_λ at the point $(\lambda_0, 0)$

$$(3.9) \quad \begin{aligned} F_\lambda(\lambda_0, 0) &= \left(\frac{d}{d\lambda} F(\lambda, 0) \right) \Big|_{\lambda=\lambda_0} \\ &= \tilde{D}'_\alpha(\lambda_0)\widehat{D}_\beta(\lambda_0) + \tilde{D}_\alpha(\lambda_0)\widehat{D}'_\beta(\lambda_0) = \tilde{D}_\alpha(\lambda_0)\widehat{D}'_\beta(\lambda_0). \end{aligned}$$

Since the eigenvalue λ_0 is simple in the spectrum of $\widehat{A}_{[\beta]}$, by Proposition 2.5(iii) $\widehat{D}'_\beta(\lambda_0) \neq 0$ holds. Similarly $\tilde{D}_\alpha(\lambda_0) \neq 0$ because of $\lambda_0 \in \rho(\tilde{A}_{[\alpha]})$. Hence we obtain that $F_\lambda(\lambda_0, 0) \neq 0$. Recall that F is C^∞ -smooth. Therefore, by the classical implicit function theorem [KP, Theorem 3.3.1] there exists the C^∞ -smooth function $\lambda(\cdot)$ defined on a sufficiently small neighborhood of the origin such that $\lambda(0) = \lambda_0$ and and that $F(\lambda(x), x) = 0$ holds pointwise. The derivative of $\lambda(\cdot)$ is given as usual by

$$(3.10) \quad \lambda'(x) = -\frac{F_x(\lambda(x), x)}{F_\lambda(\lambda(x), x)}.$$

Again using Jacobi's formula we get

$$(3.11) \quad \widehat{D}'_\beta(\lambda_0) = -\text{tr} \left(\text{adj}(\beta I_d - \widehat{M}(\lambda_0)) \widehat{M}'(\lambda_0) \right).$$

Substituting (3.8), (3.9) and (3.11) into (3.10) we arrive at $\lambda'(0) = a$ with a given by (3.3). Hence we obtain that

$$(3.12) \quad \lambda(x) = \lambda_0 + ax + O(x^2), \quad x \rightarrow 0.$$

Step III. By Proposition 2.5(ii) a point $\lambda \in \rho(\tilde{A}_0) \cap \rho(\widehat{A}_0)$ satisfying

$$\det \begin{pmatrix} \alpha I_d - \widetilde{M}(\lambda) & \omega I_d \\ \overline{\omega} I_d & \beta I_d - \widehat{M}(\lambda) \end{pmatrix} = 0$$

is in the discrete spectrum of A_Λ . By [S00, Theorem 3] one gets that

$$\det \begin{pmatrix} \alpha I_d - \widetilde{M}(\lambda) & \omega I_d \\ \overline{\omega} I_d & \beta I_d - \widehat{M}(\lambda) \end{pmatrix} = \det((\alpha I_d - \widetilde{M}(\lambda))(\beta I_d - \widehat{M}(\lambda)) - |\omega|^2 I_d).$$

That is $\lambda \in \rho(\tilde{A}_0) \cap \rho(\widehat{A}_0)$ satisfying $F(\lambda, |\omega|^2) = 0$ with F as in (3.7) belongs to the discrete spectrum of A_Λ . Hence for sufficiently small $|\omega|^2$ we have $\lambda(|\omega|^2) \in \sigma_d(A_\Lambda)$ with $\lambda(\cdot)$ defined by Step II. Finally, the expansion (3.12) implies (3.2) in the formulation of the theorem.

(ii) The proof of this item goes along the lines of the proof of (i) and we indicate only the differences. Let F be defined as in (3.7). In this special case ($d = 1$) we have

$$F(\lambda, x) = (\alpha - \widetilde{M}(\lambda))(\beta - \widehat{M}(\lambda)) - x.$$

The 1st and 2nd order partial derivatives of F are computed below

$$(3.13) \quad \begin{aligned} F_x(\lambda, x) &= -1, \quad F_{\lambda x}(\lambda, x) = 0, \quad F_{xx}(\lambda, x) = 0, \\ F_\lambda(\lambda, x) &= -\widetilde{M}'(\lambda)(\beta - \widehat{M}(\lambda)) - (\alpha - \widetilde{M}(\lambda))\widehat{M}'(\lambda), \\ F_{\lambda\lambda}(\lambda, x) &= -\widetilde{M}''(\lambda)(\beta - \widehat{M}(\lambda)) + 2\widetilde{M}'(\lambda)\widehat{M}'(\lambda) - (\alpha - \widetilde{M}(\lambda))\widehat{M}''(\lambda). \end{aligned}$$

In particular, we have at the point $(\lambda_0, 0)$

$$(3.14) \quad \begin{aligned} F_\lambda(\lambda_0, 0) &= (\widetilde{M}(\lambda_0) - \alpha)\widehat{M}'(\lambda_0), \\ F_{\lambda\lambda}(\lambda_0, 0) &= 2\widetilde{M}'(\lambda_0)\widehat{M}'(\lambda_0) + (\widetilde{M}(\lambda_0) - \alpha)\widehat{M}''(\lambda_0), \end{aligned}$$

where we used that $\beta - \widehat{M}(\lambda_0) = 0$, which is true in view of $\lambda_0 \in \sigma_d(\widehat{A}_{[\beta]})$. Similarly as on Step II in the proof of (i) we get that $F(\lambda_0, 0) = 0$ and $F_\lambda(\lambda_0, 0) \neq 0$. Hence, there exists the C^∞ -smooth function $\lambda(\cdot)$ defined on a sufficiently small neighborhood of the origin such that $\lambda(0) = \lambda_0$, that $F(\lambda(x), x) = 0$ holds pointwise and that $\lambda'(x)$ is as in (3.10). Substituting the identity $F_x(\lambda(x), x) = -1$ into (3.10) we obtain that

$$(3.15) \quad \lambda'(x) = \frac{1}{F_\lambda(\lambda(x), x)},$$

and further substituting (3.14) into the above formula we get $\lambda'(0) = a$ with a as in (3.5). Taking the derivative in (3.15) we get

$$\lambda''(x) = -\frac{F_{\lambda\lambda}(\lambda(x), x)\lambda'(x) + F_{\lambda x}(\lambda(x), x)}{(F_\lambda(\lambda(x), x))^2}.$$

Plugging (3.13) and (3.14) into the above formulae we obtain $\lambda''(0) = 2b$ with b as in (3.6). Hence we arrive at the expansion

$$\lambda(x) = \lambda_0 + ax + bx^2 + O(x^3), \quad x \rightarrow 0,$$

which implies (3.4) similarly as on Step III in the proof of (i) the expansion (3.12) implied the formula (3.2). \square

Remark 3.4. The roles of the operators $\widetilde{A}_{[\alpha]}$ and $\widehat{A}_{[\beta]}$ in the above theorem can be interchanged.

Remark 3.5. Note that $\text{adj}(0) = 1$ and in the special case $d = 1$ the formula (3.3) reduces to (3.5).

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